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The BFKL Pomeron Calculus in the dipole approach

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Abstract: In this paper we continue to pursue a goal of finding an effective theory for high energy interaction in QCD based on the colour dipole approach, for which the BFKL Pomeron Calculus gives a low energy limit. The key problem, that we try to solve in this paper is the probabilistic interpretation of the BFKL Pomeron Calculus in terms of the colourless dipoles and their interactions. We demonstrate that the BFKL Pomeron Calculus has two equivalent descriptions : (i) one is the generating functional which gives a clear probabilistic interpretation of the processes of high energy scattering and also provides a Hamiltonian-like description of the system of interacting dipoles; (ii) the second is the Langevin equation with a specific noise term which is rather complicated. We found that at high energies this Langevin equation can be reduced to the Langevin equation for directed percolation in the momentum space if the impact parameter is large, namely, $b >> 1/k$, where k is the transverse momentum of a dipole. Unfortunately, this simplified form of Langevin equation is not applicable for summation of Pomeron loops, where one integrates over all possible values of impact parameter. We show that the BFKL Pomeron calculus with two vertices (splitting $P \to P + P$ and merging $P + P \to P$ of Pomerons) can be interpreted as a system of colourless dipoles with two processes: the decay of one dipole into two and the merging of two dipoles into one dipole. However, a number of assumptions we have to make on the way to simplify the noise term in the Langevin equation and/or to apply the probabilistic interpretation, therefore, we can consider both of these approaches in the present form only as the QCD motivated models.

Keywords: BFKL Pomeron, Dipole approach, Generating functional, Semi-classical solution.

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1. Introduction

The simplest approach that we can propose for high energy interaction is based [1, 2] on the BFKL Pomeron [3] and reggeon-like diagram technique for the BFKL Pomeron interactions [4, 5, 6, 7]. This technique, which is a generalization of Gribov Reggeon Calculus [8], can be written in the elegant form of the functional integral (see [5] and the next section). It is a challenge to solve this theory in QCD finding the high energy asymptotic behaviour. However, even this simple approach has not been solved during three decades of attempts by the high energy community. This failure stimulates a search for deeper understanding of physics which is behind the BFKL Pomeron Calculus. On the other hand, it has been known for three decades that Gribov Reggeon Calculus has intrinsic difficulties [9] that are related to the overlapping of Pomerons. Indeed, due to this overlapping we have no hope that the Gribov Reggeon Calculus could be correct in describing the ultra high energy asymptotic behaviour of the amplitude. The way out of these difficulties we see in searching for a new approach which will coincide with the BFKL Pomeron Calculus at high, but not very high, energies (our correspondence principle) but it will be different in the region of ultra high energies. In a spirit of the parton approach we believe that this effective theory should be based on the interaction of 'wee' partons. We consider, as an important step in this direction, the observation that has been made at the end of the Reggeon era [10, 11, 12], that the Reggeon Calculus can be reduced to the Markov

process [13] for the probability of finding a given number of Pomerons at fixed rapidity Y. Such an interpretation, if it would be reasonable in QCD, can be useful, since it allows us to use powerful methods of statistical physics in our search of the solution.

The logic and scheme of our approach looks as follows. The first step is the Leading Log (1/x) Approximation (LLA) of perturbative QCD in which we sum all contributions of the order of $(\alpha_S \ln(1/x))^n$. In the LLA we consider such high energies that

$$\alpha_S \ln(1/x) \approx 1; \quad \text{while} \quad \alpha_S \ll 1 \tag{1.1}$$

It is well known that the LLA approach generates the BFKL Pomeron (see [3] and the next section) which leads to the power-like increase of the scattering amplitude ($A \propto \frac{1}{x^{\omega(n=0,\nu=0)}}$ with $\omega(n=0,\nu=0) \propto \alpha_S$).

The second step is the BFKL Pomeron Calculus in which we sum all contributions of the order of

$$\left(\frac{\bar{\alpha}_S^2}{N_c^2}\frac{1}{x^{\omega(n=0,\nu=0)}}\right)^n \quad \text{therefore} \quad \frac{\bar{\alpha}_S^2}{N_c^2}\frac{1}{x^{\omega(n=0,\nu=0)}} \approx 1 \tag{1.2}$$

where $\bar{\alpha}_S = \alpha_S N_c/\pi$.

The structure of this approach as well as its parameter has been understood before QCD [14] and was confirmed in QCD (see [1, 2, 4, 5, 6, 7, 15, 16]). This calculus extends the region of energies from $\ln(1/x) \approx 1/\bar{\alpha}_S$ of LLA to $\ln(1/x) \approx (1/\bar{\alpha}_S)\ln(N_c^2/\bar{\alpha}_S)$. The BFKL Pomeron Calculus describes correctly the scattering process in the region of energy:

$$\frac{1}{\bar{\alpha}_S}\ln\left(\frac{N_c^2}{\bar{\alpha}_S}\right) \ll ln\frac{1}{x} \ll \frac{1}{\bar{\alpha}_S^2} \tag{1.3}$$

For higher energies the corrections of the order of $\left(\bar{\alpha}_S^2 \ln(1/x)\right)^n$ should be taken into account making all calculations very complicated.

Our credo is that we will be able to describe the high energy processes outside of the region of Eq. (1.3), if we could find an effective theory which describes the BFKL Pomeron calculus in the kinematic region given by Eq. (1.3), but based on the microscopic degrees of freedom and not on the BFKL Pomeron. In so doing, we hope that we can avoid all intrinsic difficulties of the BFKL Pomeron calculus and build an approximation that will be in an agreement with all general theorems like the Froissart bounds and so on. Solving this theory, we can create a basis for moving forward considering all corrections to this theory due to higher orders in $\bar{\alpha}_S$ contributions, running QCD coupling and others.

The goal of this paper is to consider the key problem: the probabilistic interpretation of the BFKL Pomeron Calculus based on the idea that colourless dipoles are the correct degrees of freedom in high energy QCD [17]. We believe that colourless dipoles and their interaction will lead to a future theory at high energies which will have the BFKL Pomeron Calculus as the low energy limit (see Eq. (1.3)) and which will allow us to avoid all difficulties of dealing with BFKL Pomerons at ultra high energies.

Colourless dipoles play two different roles in our approach. First, they are partons ('wee' partons) for the BFKL Pomeron. This role is not related to the large N_c approximation and, in principle, we can always speak about probability to find a definite number of dipoles instead of defining the probability to find a number of the BFKL Pomerons. The second role of the colour dipoles is that at high energies we can interpret the vertices of Pomeron merging and splitting in terms of probability for two dipoles to merge into one dipole and of probability for decay of one dipole into two ones. It was shown in [17] that $P \to 2P$ splitting can be described as the process of the dipole decay into two dipoles. However, the relation between the Pomeron merging ($2P \to P$) and the process of merging of two dipoles into one dipole is not so obvious and it will be discussed here.

This paper is a next step in our programme of searching the simplest but correct approach to high energy scattering in QCD in which we continue the line of thinking presented in [18, 19, 20, 21]. The outline of the paper looks as follows.

In the next section we will discuss the BFKL Pomeron Calculus in the elegant form of the functional integral, suggested by M. Braun about five years ago [5]. In the framework of this approach we find a set of equations for the amplitude of n-dipole interaction with the target. We show that the recent intensive work on this subject [22, 23, 20] confirms the BFKL Pomeron Calculus in spite of the fact that these attempts were based on slightly different assumptions.

In section three we demonstrate statistical interpretation of the theory with interacting Pomerons. The one-to-one correspondence between BFKL Pomeron calculus and Langevin theory is found by showing how full Lagrangian generates stochastic Langevin equation with a peculiar noise term. In toy model with zero transverse dimensions this noise is reduced to one typical for directed percolation. Unfortunately the complexity of the noise term restricts the practical use of Langevin equation in this form, and one should look for further simplifications. One of them is to assume that impact parameter is much larger than any dipole size in the system (see Eq. (3.20)). Using this assumption and going to momentum space we rewrite our theory in the form of Langevin equation with a noise term proportional to the field (directed percolation universality class). It should be mentioned that large impact parameter approximation is unapplicable for summation of Pomeron loops, where we integrate over all possible values of the impact parameter.

Next, we discuss an approach based on generating functional. We show the equivalence between generating functional approach and the BFKL Pomeron calculus in the kinematical region Eq. (1.3) that leads to a clear interpretation of the BFKL Pomeron calculus as an alternative description of the system of interacting colourless dipoles. The interrelation between vertices of the Pomeron interactions and the microscopic dipole processes is considered. It is instructive to notice that the generating functional approach leads to a feedback to the BFKL Pomeron Calculus restricting the integration over Pomeron fields in the functional integral by the range $0 < \Phi < 1/4\pi\alpha_S$.

In the fourth section we suggest a practical way of building the Monte Carlo code to solve the equation for the generating functional which can be a basis for consideration of the multiparticle production processes.

In conclusion we are going to compare our approach with other approaches on the market.

2. The BFKL Pomeron Calculus

2.1 The general structure of the BFKL Pomeron calculus

We start with a general structure of the BFKL Pomeron calculus in QCD. The BFKL Pomeron exchange can be written in the form (see Fig. 1-1)

$$A(\text{Fig. 1-1}) \;=\; V_u \bigotimes G_P(r_1, r_2; b|Y_1 - Y_2) \bigotimes V_d \;\propto\; \frac{\bar{\alpha}_S^2}{N_c^2} \exp\left(\omega(n=0,\nu=0)\,(Y_1 - Y_2)\right) \tag{2.1}$$

with $\omega(n=0,\nu=0) \propto \bar{\alpha}_S$,$Y_1 - Y_2 = \ln(1/x)$ and $\bigotimes$ denotes the all needed integrations.

It is easy to understand the main parameters of the BFKL Pomeron calculus by comparing the contributions of the first 'fan' diagrams of Fig. 1-2 with the one BFKL Pomeron exchange.

This diagram has the following contribution

$$\begin{aligned} A(\text{Fig. 1-2}) \;&=\; \int_{Y_2}^{Y_1} dY'\, V_u \bigotimes G_P(r_1, r'; b|Y_1 - Y') \bigotimes \Gamma(1\to 2)\, G_P^2(r', r_2; b|Y' - Y_2) \bigotimes V_d^2 \\ &\propto \frac{V_u\, V_d^2\, \Gamma(1\to 2)}{\omega(n=0,\nu=0)} \exp\left(2\,\omega(n=0,\nu=0)\,(Y_1 - Y_2)\right) \;\propto\; \left(\frac{\bar{\alpha}_S^2}{N_c^2}\right)^2 \exp\left(2\,\omega(n=0,\nu=0)\,Y_1 - Y_2\right) \end{aligned} \tag{2.2}$$

where r_1 and r_2 are the sizes of the projectile and target dipoles while $\{r'\}$ denotes all dipole variables in Pomeron splitting and/or merging.

One can see that the ratio of this two diagrams is proportional to $\frac{\bar{\alpha}_S^2}{N_c^2}\exp\left(\omega(n=0,\nu=0)(Y_1-Y_2)\right)$ which is the parameter given by Eq. (1.2). When this ration is about 1 we need to calculate all diagrams with the Pomeron exchange and their interactions (see Fig. 1-a - Fig. 1-f). All vertices, that are shown in Fig. 1, has been calculated in [4, 5] and they have the following order in $\bar{\alpha}_S$ [1]:

$$\omega(n=0,\nu=o) \;\propto\; \bar{\alpha}_S\;;\quad \Gamma(1\to 2) \;\propto\; \frac{\bar{\alpha}_S^2}{N_c}\;;\quad \Gamma(2\to 1) \;\propto\; \frac{\bar{\alpha}_S^2}{N_c}\;;$$

$$\Gamma(2\to 2) \;\propto\; \frac{\bar{\alpha}_S}{N_c^2}\;;\Gamma(2\to 3) \;\propto\; \frac{\bar{\alpha}_S^2}{N_c^2}\;;V_u \;\propto\; \frac{\bar{\alpha}_S}{N_c}\;;V_d \;\propto\; \frac{\bar{\alpha}_S}{N_c}\;; \tag{2.3}$$

Using Eq. (2.3) we can easily estimate the contributions of all diagrams in Fig. 1. Namely,

$$A(\text{Fig. 1-3}) \;\propto\; L^2(Y)\;;\quad A(\text{Fig. 1-4}) \;\propto\; \frac{\bar{\alpha}_S}{N_c^2}\,Y\,L^2(Y)\;; \tag{2.4}$$

$$A(\text{Fig. 1-5}) \;\propto\; \frac{1}{N_c^2}\,L^3(Y)\,;\quad A(\text{Fig. 1-6}) \;\propto\; \frac{\bar{\alpha}_S}{N_c^2}\,Y\,L^2(Y)\,; \tag{2.5}$$

with $L(Y) \;=\; \left(\bar{\alpha}_S^2/N_c^2\right)\,\exp\left(\omega(n=0,\nu=0)\,Y\right)$.

As we have mentioned in the introduction the BFKL calculus sums all diagrams at such a high energy that parameter $L(Y)$ is of the order of 1 (see Eq. (1.2)). In this kinematic region we need to take into account the diagrams of Fig. 1-1 , Fig. 1-2 and Fig. 1-3 (see Eq. (2.1),Eq. (2.2) and Eq. (2.4)). Indeed, diagrams of Fig. 1-4 and Fig. 1-6 (see Eq. (2.4) and Eq. (2.5)) are small since $(\bar{\alpha}_S^2/N_c^2)\,Y \;\ll\; 1/N_c^2 \;\ll\; 1$ in the kinematic region of Eq. (1.3), while the diagrams of Fig. 1-5 (see Eq. (2.5)) are small at $L(Y)\approx 1$.

The first conclusion that we can derive from this analysis that in the kinematic region where $L(Y) \;\approx\; 1$ we need to take into account all diagrams with $\Gamma(1\to 2)$ and $\Gamma(2\to 1)$ vertices while the diagrams with $\Gamma(2\to 2)$ and $\Gamma(2\to 3)$ vertices give small, negligible contributions.

However, if $L(Y) \;\propto\; N_c$ one can see from Eq. (2.1) - Eq. (2.5) that all diagrams give so essential contributions that we have to take them into account. Indeed, for such $L(Y)$ $A(\text{Fig. 1-3}) \;\propto\; N_c^2$, $A(\text{Fig. 1-4}) \;\propto\; \bar{\alpha}_S Y \;>\; 1$,$A(\text{Fig. 1-5}) \;\propto\; N_c^2$ and $A(\text{Fig. 1-6}) \;\propto\; \bar{\alpha}_S Y \;>\; 1$.

It is interesting to notice that the vertex $\Gamma(2\to 2) \;\propto\; \bar{\alpha}_S^4/N_c^2$ can be neglected even at such large values of $L(Y)$.

Finally, we can conclude that the first step of our approach can be summing of the diagrams with $\Gamma(1\to 2)$ and $\Gamma(2\to 1)$ vertices in the kinematic region $L(Y)\approx 1$ or $Y \;\;1/\bar{\alpha}_S\,\ln\left(N_c^2/\bar{\alpha}_S\right)$.

However, we would like to stress that we need to make an additional assumption inherent for the BFKL Pomeron calculus: the multi-gluon states in t-channel of the scattering amplitude lead to smaller contribution at high energies than the exchange of the correspondent number of the BFKL Pomerons (see more in [21]). This statement is supported by the fact that numerous attempts to find the intercept of these states being larger than the intercept for multi-Pomeron exchanges[25] have failed.

2.2 The path integral formulation of the BFKL calculus

The main ingredient of the BFKL Pomeron calculus is the Green function of the BFKL Pomeron describing the propagation of a pair of gluons from rapidity Y' and points x'_1 and x'_2 to rapidity Y and points x_1 and x_2 [2]. Since

[1] In Eq. (2.3) we use the normalizations of these vertices which are originated from calculation of the Feynman diagrams. In the dipole approach we use a different normalization (see below section 3 and 4) but all conclusions do not depend on the normalization.

[2] Coordinates x_i here are two dimensional vectors and, strictly speaking, should be denoted by $\vec{x}_i$ or $\mathbf{x_i}$. However, we will use notation x_i hoping that it will not cause difficulties in understanding.

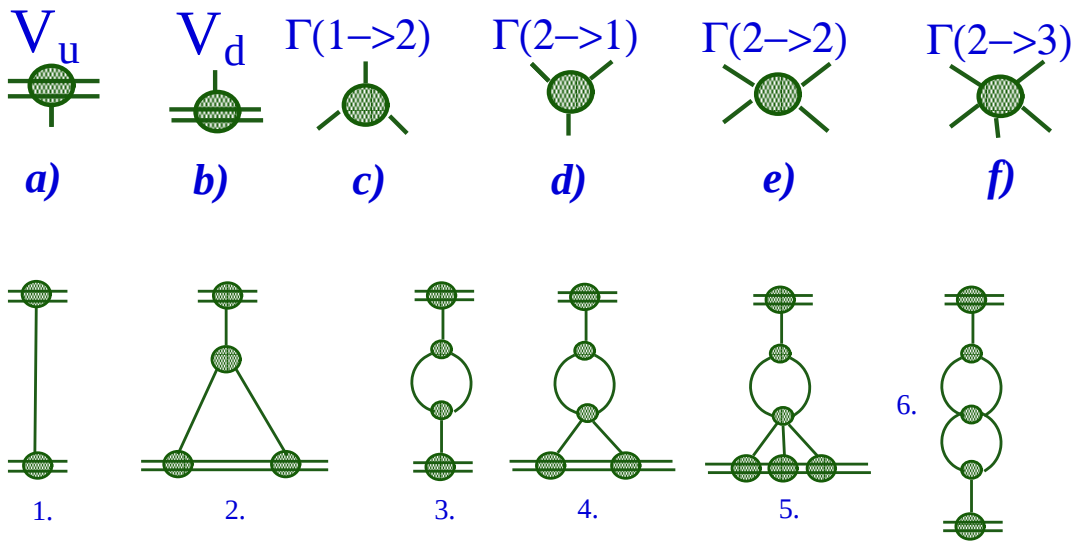

Figure 1: The BFKL Pomeron interactions and the examples of the diagrams of the BFKL Pomeron calculus in QCD. The solid line describes the Pomeron exchange while the double line stands for the dipole.

the Pomeron does not carry colour in t-channel we can treat initial and final coordinates as coordinates of quark and antiquark in a colourless dipole. This Green function is well known[26], and has a form

$$G(x_1', x_2'; Y | x_1, x_2; Y') \;=\; \Theta(Y - Y') \times \tag{2.6}$$

$$\times \sum_{n=-\infty}^{\infty} \int d\nu \; d^2x_0 \, e^{\omega(n,\nu)(Y-Y')} \, \lambda(n,\nu) \, E_n(x_1', x_2'; x_0|\nu) \, E_n^*(x_1, x_2; x_0|\nu)$$

where vertices E_n are given by

$$E_n(x_1, x_2; x_0|\nu) \;=\; \left(\frac{x_{12}}{x_{10}\, x_{20}}\right)^h \left(\frac{x_{12}^*}{x_{10}^*\, x_{20}^*}\right)^{\tilde{h}} \tag{2.7}$$

where $x_{ik} = x_i - x_k$, $x_i = x_{i,x} + i x_{i,y}$ [3] ,$x_i^* = x_{i,x} + i x_{i,y}$; $h = (1-n)/2 + i\nu$ and $\tilde{h} = 1 - h^*$. The energy levels $\omega(n,\nu)$ are the BFKL eigen values

$$\omega(n,\nu) \;=\; \bar{\alpha}_S \left(\psi(1) - Re\, \psi\left(\frac{|n|+1}{2} + i\nu \right) \right) \tag{2.8}$$

where $\psi(z) = d \ln \Gamma(z)/dz$ and $\Gamma(z)$ is the Euler gamma function. Finally

$$\lambda(n,\nu) \;=\; \frac{1}{[(n+1)^2 + 4\nu^2][(n-1)^2 + 4\nu^2]} \tag{2.9}$$

The interaction between Pomerons is depicted in Fig. 2 and described by the triple Pomeron vertex which can be written in the coordinate representation [5] for the following process: two gluons with coordinates x_1' and x_2' at

[3] $x_{i,x}$ and $x_{i,y}$ are components of the two dimensional vector x_i on x-axis and y- axis

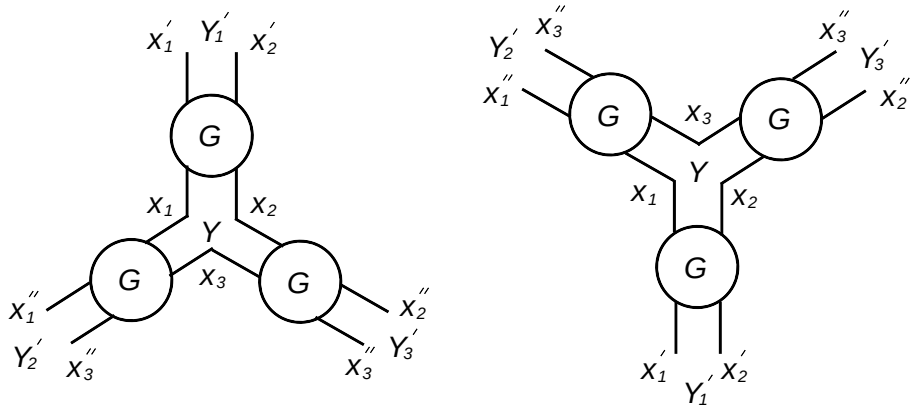

Figure 2: The graphic form of the triple Pomeron vertex in the coordinate representation.

rapidity Y_1' decay into two gluon pairs with coordinates x_1'' and x_3'' at rapidity Y_2' and x_2'' and x_3'' at rapidity Y_3' due to the Pomeron splitting at rapidity Y. It looks as

$$2\frac{\pi\,\bar{\alpha}_S^2}{N_c}\int\frac{d^2x_1\,d^2\,x_2\,d^2x_3}{x_{12}^2\,x_{23}^2\,x_{13}^2}\left(G(x_1',x_2';Y_1'|x_1,x_2;Y)\overleftarrow{L}_{1,2}\right)G(x_1,x_3;Y|x_1'',x_3'';Y_2')\,G(x_3,x_2;Y|x_3'',x_2'';Y_3') \tag{2.10}$$

where

$$\overleftarrow{L}_{1,2}\;=\;r_{12}^4\,p_1^2\,p_2^2\;\text{ with }p^2\;=\;-\,\nabla^2 \tag{2.11}$$

and arrow shows the direction of action of the operator L. For the inverse process of merging of two Pomerons into one we have

$$2\frac{\pi\,\bar{\alpha}_S^2}{N_c}\int\frac{d^2x_1\,d^2\,x_2\,d^2x_3}{x_{12}^2\,x_{23}^2\,x_{13}^2}\,G(x_1'',x_3'';Y_2'|x_1,x_3;Y)\,G(x_3'',x_2'';Y_3'|x_3,x_2;Y)\,\left(\overrightarrow{L}_{1,2}\,G(x_1,x_2;Y|x_1',x_2';Y_1')\right) \tag{2.12}$$

The theory with the interaction given by Eq. (2.10) and Eq. (2.12) can be written through the functional integral as was proposed and developed by Braun in [5]. We include a discussion of the basics of this approach for the sake of completeness of our presentation.

$$Z[\Phi,\Phi^+]\;=\;\int\,D\Phi\,D\Phi^+\,e^S\;\text{ with}\,S\;=\;S_0\;+\;S_I\;+\;S_E \tag{2.13}$$

where S_0 describes free Pomerons, S_I corresponds to their mutual interaction while S_E relates to the interaction with external sources (target and projectile). From Eq. (2.10) and Eq. (2.12) it is clear that

$$S_0\;=\;\int dY\,dY'\,d^2x_1\,d^2x_2\,d^2x_1'\,d^2x_2'\,\Phi^+(x_1,x_2;Y)\,G^{-1}(x_1,x_2;Y|x_1',x_2';Y')\,\Phi(x_1',x_2';Y') \tag{2.14}$$

$$S_I\;=\;\frac{2\,\pi\bar{\alpha}_S^2}{N_c}\int dY\int\frac{d^2x_1\,d^2x_2\,d^2x_3}{x_{12}^2\,x_{23}^2\,x_{13}^2}\left\{\left(\overrightarrow{L}_{1,2}\Phi(x_1,x_2;Y)\right)\;\Phi^+(x_1,x_3;Y)\,\Phi^+(x_3,x_2;Y)\;+\;h.c.\right\} \tag{2.15}$$

For S_E we have local interaction both in rapidity and in coordinates, namely,

$$S_E\;=\;-\int dY\,d^2x_1\,d^2x_2\left\{\Phi(x_1,x_2;Y)\,\tau_{pr}(x_1,x_2;Y)\;+\;\Phi^+(x_1,x_2;Y)\,\tau_{tar}(x_1,x_2;Y)\right\} \tag{2.16}$$

where τ_{pr} (τ_{tar}) stands for the projectile and target, respectively. The form of functions τ depend on the non-perturbative input in our problem and for the case of nucleus target they are given in [5].

For the case of projectile being a dipole that scatters off a nucleus the scattering amplitude has the form

$$T(x_1,y_1;Y)\;\equiv T^{(1)}(x_1,y_1;Y)\;=\;-\frac{4\,\pi^2\,\bar{\alpha}_S}{N_c}\,\frac{\int\,D\Phi\,D\Phi^+\,\Phi(x_1,y_1;Y)\,e^{S[\Phi,\Phi^+]}}{\int\,D\Phi\,D\Phi^+\,e^{S[\Phi,\Phi^+]}|_{S_E=0}} \tag{2.17}$$

where extra α_S comes from our normalization and we neglect term with τ_{pr} in Eq. (2.16).

Generally, for the amplitude of interaction of n dipoles at rapidity Y we can write the following expression [4]

$$T^{(n)}(x_1,y_1,\ldots x_n,y_n;Y)\;=\;(-1)^n\left(\frac{4\,\pi^2\,\bar{\alpha}_S}{N_c}\right)^n\,\frac{\int\,D\Phi\,D\Phi^+\,\prod_{i=1}^n\Phi(x_i,y_i;Y)\;e^{S[\Phi,\Phi^+]}}{\int\,D\Phi\,D\Phi^+\,e^{S[\Phi,\Phi^+]}|_{S_E=0}} \tag{2.18}$$

The extra factor $(-1)^n$ is due to the fact that in S_E the source for both projectile and target, has extra minus sign.

It is useful to introduce the Green function of the BFKL Pomeron that includes the Pomeron loops. This function has the form

$$G\,(x_1,y_1;Y|x_2,y_2;Y')\;=\;\frac{\int\,D\Phi\,D\Phi^+\,\Phi^+(x_1,y_1;Y)\,\Phi(x_2,y_2;Y')\;e^{S[\Phi,\Phi^+]}}{\int\,D\Phi\,D\Phi^+\,e^{S[\Phi,\Phi^+]}|_{S_E=0}} \tag{2.19}$$

For further presentation we need some properties of the BFKL Green function [26]:

1. Generally,

$$G^{-1}(x_1,x_2;Y|x'_1,x'_2;Y')\;=\;p_1^2\,p_2^2\,\left(\frac{\partial}{\partial Y}+H\right)\;=\;\left(\frac{\partial}{\partial Y}+H^+\right)\,p_1^2\,p_2^2; \tag{2.20}$$

$$Hf(x_1,x_2;Y)\;=\;\frac{\bar{\alpha}_S}{2\pi}\int\frac{d^2x_3\,x_{12}^2}{x_{23}^2\,x_{13}^2}\;(f(x_1,x_2;Y)\;-\;f(x_1,x_3;Y)\;-\;f(x_3,x_2;Y))\,; \tag{2.21}$$

2. The initial Green function (G_0) is equal to

$$G_0(x_1,x_2;Y|x'_1,x'_2;Y)\;=\;\pi^2\,\ln\frac{x_{1,1'}^2\,x_{2,2'}^2}{x_{1,2'}^2\,x_{1',2}^2}\,\ln\frac{x_{1,1'}^2\,x_{2,2'}^2}{x_{1,2}^2\,x_{1',2'}^2} \tag{2.22}$$

This form of G_0 has been discussed in [26]. In appendix A we demonstrate that this expression for $G_0(x_1,x_2;Y|x'_1,x'_2;Y)$ stems from $\omega=\omega(n=0,\nu)$ term in sum of Eq. (2.6). Only this term is essential at high energies since all other terms lead to contributions decreasing with energy.

3. It should be stressed that

$$\nabla_1^2\,\nabla_2^2\,G_0(x_1,x_2;Y|x'_1,x'_2;Y)\;=\;(2\,\pi)^4\;\left(\delta^{(2)}(x_1-x'_1)\,\delta^{(2)}(x_2-x'_2)\;+\;\delta^{(2)}(x_1-x'_2)\,\delta^{(2)}(x_2-x'_1)\right) \tag{2.23}$$

4. In the sum of Eq. (2.6) only the term with $n=0$ is essential for high energy asymptotic behaviour since all $\omega(n,\nu)$ with $n\geq 1$ are negative and, therefore, lead to contributions that decrease with energy. Taking into account only the first term one can see that G is the eigen function of operator L_{13}, namely

$$L_{12}\,G(x_1,x_2;Y|x'_1,x'_2;Y')\;=\;\frac{1}{\lambda(0,\nu)}\,G(x_1,x_2;Y|x'_1,x'_2;Y')\;\approx\;G(x_1,x_2;Y|x'_1,x'_2;Y'); \tag{2.24}$$

[4] Starting from this equation we use notations x_i for the coordinates of quark while y_i denote the coordinates of antiquarks. For rapidity we will use Y.

The last equation holds only approximately in the region where $\nu \ll 1$, but this is the most interesting region which is responsible for high energy asymptotic behaviour of the scattering amplitude.

All properties of the BFKL Pomeron Green function as well as of the functional integral approach to the BFKL Pomeron calculus have been discussed (for more information see [5, 6, 26]).

In the next section we will derive the chain of equations for multi-dipole amplitude in the BFKL Pomeron calculus and will show that these equations are the same as ones that have been discussed in framework of dipole approach [22, 23, 20].

2.3 The chain of equations for the multi-dipole amplitudes

Using Eq. (2.13) and Eq. (2.17) we can easily obtain the chain equation for multi-dipole amplitude $T^{(n)}$ noticing that every dipole interacts only with one Pomeron (see Eq. (2.17)).

These equations follow from the fact that a change of variables does not alter the value of functional integral of Eq. (2.13). In particular, $Z[\Phi,\Phi^+] = Z[\Phi,\Phi'^+]$ (see Eq. (2.13)) where $\Phi'^+ = \Phi^+ + \epsilon(x,y)$ with a small function $\epsilon(x,y)$. Therefore,

$$\int D\,\Phi\, D\,\Phi^+\, e^{S[\Phi,\Phi^+]} \;=\; \int D\,\Phi\, D\,\Phi'^+\, e^{S[\Phi,\Phi'^+]} \tag{2.25}$$

Substituting $\Phi'^+ = \Phi^+ + \epsilon(x,y)$ and expanding this equation to first order in ϵ, we find

$$0 \;=\; \int D\,\Phi\, D\,\Phi^+\, e^{S[\Phi,\Phi^+]} \times \tag{2.26}$$

$$[\int dY\, dY'\, d^2x_1\, d^2x_2\, d^2x'_1\, d^2x'_2\, \epsilon(x_1,x_2;Y)\, G^{-1}(x_1,x_2;Y|x'_1,x'_2;Y')\, \Phi(x'_1,x'_2;Y')$$

$$+\frac{2\,\pi\bar{\alpha}_S^2}{N_c}\int dY \int \frac{d^2x_1\, d^2x_2\, d^2x_3}{x_{12}^2\, x_{23}^2\, x_{13}^2}\cdot\{\left(\epsilon(x_1,x_2;Y)\hat{L}_{1,2}\right)\cdot\Phi(x_1,x_3;Y)\,\Phi(x_3,x_2;Y)+$$

$$2\left(\hat{L}_{1,2}\Phi(x_1,x_2;Y)\right)\cdot\epsilon(x_1,x_3;Y)\,\Phi^+(x_3,x_2;Y)\} - \int dY\, d^2x_1\, d^2x_2\,\epsilon(x_1,x_2;Y)\,\tau_{tar}(x_1,x_2;Y)]$$

We redefine the integration variables in the third term as follows

$$2\left(\hat{L}_{1,2}\Phi(x_1,x_2;Y)\right)\cdot\epsilon(x_1,x_3;Y)\,\Phi^+(x_3,x_2;Y) \longrightarrow 2\left(\hat{L}_{1,3}\Phi(x_1,x_3;Y)\right)\cdot\epsilon(x_1,x_2;Y)\,\Phi^+(x_2,x_3;Y)$$

Using the expression for the Hamiltonian Eq. 2.20 and the Casimir operator Eq. 2.11 we define a new variation parameter $\epsilon(x_1,x_2;Y)p_1^2\,p_2^2$. In terms of this variation parameter Eq. 2.26 reads as

$$0 \;=\; \int D\,\Phi\, D\,\Phi^+\, e^{S[\Phi,\Phi^+]} \times \tag{2.27}$$

$$[\int dY\, d^2x_1\, d^2x_2\;\; \epsilon(x_1,x_2;Y)p_1^2p_2^2\left(\frac{\partial}{\partial Y}+H\right)\Phi(x_1,x_2;Y)$$

$$+\frac{2\,\pi\bar{\alpha}_S^2}{N_c}\int dY \int \frac{d^2x_1\, d^2x_2\, d^2x_3}{x_{12}^2\, x_{23}^2\, x_{13}^2}\cdot\{\left(\epsilon(x_1,x_2;Y)p_1^2p_2^2\right)\cdot\Phi(x_1,x_3;Y)\,\Phi(x_3,x_2;Y)+$$

$$2\left(\hat{L}_{1,3}\Phi(x_1,x_3;Y)\right)\cdot\epsilon(x_1,x_2;Y)\,\frac{p_1^2p_2^2}{p_1^2p_2^2}\,\Phi^+(x_3,x_2;Y)\} - \int dY\, d^2x_1\, d^2x_2\,\epsilon(x_1,x_2;Y)\frac{p_1^2p_2^2}{p_1^2p_2^2}\,\tau_{tar}(x_1,x_2;Y)]$$

We denote the new variation parameter by $\tilde{\epsilon}(x_1,x_2;Y) = p_1^2p_2^2\epsilon(x_1,x_2;Y)$ and use the property of the initial Green function Eq. 2.23 to rewrite $\frac{1}{p_1^2p_2^2}$ in terms of G_0 as follows

$$\frac{1}{p_1^2p_2^2}\tilde{\epsilon}(x_1,x_2;Y) \;=\; \frac{1}{2\,(2\pi)^4}\int G_0(x_1,x_2;Y|x'_1,x'_2;Y)\tilde{\epsilon}(x'_1,x'_2;Y)d^2x'_1d^2x'_2 \tag{2.28}$$

Thus, Eq. 2.27 can be written as

$$0 \;=\; \int D\,\Phi\,D\,\Phi^{+}\,e^{S[\Phi\,,\Phi^{+}]}\,\times \tag{2.29}$$

$$[\int dY\,d^2x_1\,d^2x_2\;\;\tilde{\epsilon}(x_1,x_2;Y)\left(\frac{\partial}{\partial Y}+H\right)\Phi(x_1,x_2;Y)$$

$$+\frac{2\,\pi\bar{\alpha}_S^2}{N_c}\int dY\int\frac{d^2x_1\,d^2x_2\,d^2x_3}{x_{12}^2\,x_{23}^2\,x_{13}^2}\cdot\{\tilde{\epsilon}(x_1,x_2;Y)\;\Phi(x_1,x_3;Y)\,\Phi(x_3,x_2;Y)+$$

$$2\left(\hat{I\!\!E}_{1,3}\Phi(x_1,x_3;Y)\right)\cdot\left\{\frac{1}{2\,(2\pi)^4}\int G_0(x_1,x_2;Y|x_1',x_2';Y)\tilde{\epsilon}(x_1',x_2';Y)d^2x_1'd^2x_2'\right\}\Phi^{+}(x_3,x_2;Y)\}$$

$$-\int dY\,d^2x_1\,d^2x_2\left\{\frac{1}{2\,(2\pi)^4}\int G_0(x_1,x_2;Y|x_1',x_2';Y)\tilde{\epsilon}(x_1',x_2';Y)d^2x_1'd^2x_2'\right\}\tau_{tar}(x_1,x_2;Y)]$$

Noting that the r.h.s. of Eq. (2.29) should vanish for any possible variation of $\tilde{\epsilon}(x_1,x_2;Y)$ we obtain

$$0 \;=\; \int D\,\Phi\,D\,\Phi^{+}\,e^{S[\Phi\,,\Phi^{+}]}\,\times[\left(\frac{\partial}{\partial Y}+H\right)\Phi(x_1,x_2;Y) \tag{2.30}$$

$$+\frac{2\,\pi\bar{\alpha}_S^2}{N_c}\;\int\frac{x_{12}^2\,d^2x_3}{x_{23}^2\,x_{13}^2}\Phi(x_1,x_3;Y)\,\Phi(x_3,x_2;Y)+$$

$$+\frac{2\,\pi\bar{\alpha}_S^2}{N_c}\,\frac{2}{(2\pi)^4}\int\frac{d^2x_1'\,d^2x_2'\,d^2x_3}{x_{1'2'}^2x_{2'3}^2\,x_{1'3}^2}\left(x_{1'3}^4\,p_{1'}^2p_3^2\Phi(x_1',x_3;Y)\right)G_0(x_1',x_2';Y|x_1,x_2;Y)\,\Phi^{+}(x_3,x_2';Y)\}$$

$$-\frac{1}{2\,(2\pi)^4}\int d^2x_1'd^2x_2'G_0(x_1',x_2';Y|x_1,x_2;Y)\,\tau_{tar}(x_1',x_2';Y)]$$

where we interchanged $(x_1',x_2')\leftrightarrow(x_1,x_2)$. We notice that the third and last terms are independent of rapidity and can be absorbed in the initial condition. This is obvious for the last term which represents the target source. To show this for the third term we use the property of the Casimir operator at high energies ($n=0\;,\;\nu=0$)

$$\hat{I\!\!E}_{1,3}\Phi(x_1,x_3;Y)\simeq\Phi(x_1,x_3;Y)$$

and the definition of the Green function (see Eq. 2.19). This equation will be discussed in the next section in more details. We see that the third term results into the product of two initial Green functions which are independent of rapidity.

Now we can use the definition of the amplitude defined in Eq. 2.17 and Eq. 2.18 to rewrite Eq. 2.30 in a simple form

$$\frac{\partial T^{(1)}(x_1,x_2;Y)}{\partial Y}\;=\;\frac{\bar{\alpha}_S}{2\,\pi}\int d^2\,z\,K(x_1,x_2;z) \tag{2.31}$$

$$\left(T^{(1)}(x_1,z;Y)\;+\;T^{(1)}(z,x_2;Y)\;-\;T^{(1)}(x_1,x_2;Y)\;-\;T^{(2)}(Y;x_1,z;z,x_2;Y)\right)$$

where kernel $K(x,y|z)$ is defined as

$$K(x,y|z)\;=\;\frac{(x-y)^2}{(x-z)^2\,(z-y)^2} \tag{2.32}$$

and the Hamiltonian is given by Eq. 2.21.

This equation has a very simple meaning that is clear from Fig. 3.

Starting from equation

$$\int D\,\Phi\,D\,\Phi^{+}\,\Phi(Y;x_4,x_5)\;e^{S[\Phi\,,\Phi^{+}]}\;=\;\int D\,\Phi\,D\,\Phi'^{+}\,\Phi(Y;x_4,x_5)\;e^{S[\Phi\,,\Phi'^{+}]} \tag{2.33}$$

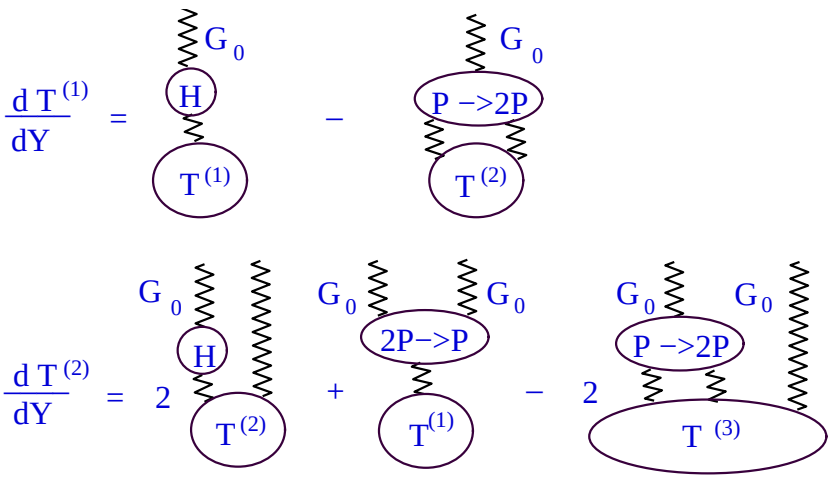

Figure 3: The graphic form of equations for the multi dipole amplitude.

we obtain the equation for the amplitude $T^{(2)}$, namely,

$$\frac{\partial T^{(2)}(x_1,x_2;x_3,x_4;Y)}{\partial Y} \;=\; \frac{\bar{\alpha}_S}{2\,\pi}\int d^2\,z\,K(x_1,x_2|z) \tag{2.34}$$

$$\Big(T^{(2)}(x_1,z;x_3,x_4;Y)\;+\;T^{(2)}(z,x_2;x_3,x_4;Y)\;-\;T^{(2)}(x_1,x_2;x_3,x_4;Y)\,-$$

$$-\,T^{(3)}(x_1,z;z,x_2;x_3,x_4;Y)\Big)\;+$$

$$+\,\frac{\bar{\alpha}_S^2}{N_c^2}\,\frac{\bar{\alpha}_S}{2\,\pi}\int d^2\,x'\,d^2\,x''\,\Gamma_{2\to 1}(x_1,x_2;x_3,x_4|x',x'')\,\nabla^2_{x'}\nabla^2_{x''}T^{(1)}(x',x'';Y);$$

where function $\Gamma_{2\to 1}$ is equal to

$$\Gamma_{2\to 1}(x_1,y_1;x_2,y_2|x,y)\;=\int d^2z K(x,y;z)\;G_0(x_1,y_1|x,z)\,G_0(x_1,y_1|z,y) \tag{2.35}$$

In derivation of Eq. (2.31) and Eq. (2.34) we use Eq. (2.22) and Eq. (2.23) as well as normalization condition (see Eq. (2.17)) for the scattering amplitude. These two equations are the same as in [27]. This shows that approach developed in [22, 27] gives a result similar to that obtained from BFKL Pomeron Calculus.

Assuming $T^{(2)} = T^{(1)}\,T^{(1)}$ we obtain the Balitsky-Kovchegov equation [28, 29]. We can do this only if we can argue why the Pomeron splitting is more important than the Pomeron merging. For example, this assumption is reasonable for scattering of the dipole with the nucleus target. Generally speaking, the splitting and merging have the same order in α_S (see Eq. (2.10) and Eq. (2.12)). In Eq. (2.31) and Eq. (2.34) these two processes look like having a different order of magnitude in α_S, but this fact does not interrelate with any physics and reflects only our normalization. However, we will see that for a probabilistic interpretation the correct normalization is very important.

3. Generating functional and probabilistic interpretation

3.1 Statistical physics analogy: Langevin equation and directed percolation

The functional of Eq. (2.13) is reminiscent of the partition function of statistical mechanics. Indeed, the partition function has a general form

$$Z[H] \;=\; e^{-\frac{1}{kT}\,F[H]} \;=\; \int d\mathbf{s}\, e^{-\frac{1}{kT}\,\int\, dx(\mathcal{H}(s)\,-H\,s(x))} \tag{3.1}$$

where $F[H]$ is the Helmholtz free energy. As an example, Eq. (3.1) is written for the system of spins with the energy density $\mathcal{H}(s)$ in the external magnetic field H. The integration is performed over all possible spin configuration in the system.

Comparing Eq. (3.1) and Eq. (2.13) one can see that Eq. (2.13) describes a statistical system with $kT = 1$ and with $F[-\tau_{tr}] = -S$. The form of S_E suggests that $H = -\tau_{tr}$ plays a role of the external field.

We demonstrate the idea of the statistical interpretation of our theory by showing how one can generate full interaction Lagrangian starting with Langevin equation with a specific choice of a noise term.

We define a partition function as an integral over all possible field configurations being restricted to satisfy Langevin equation with a Gaussian noise

$$Z[\Phi,\zeta] \sim \int P(\zeta)\; D\zeta\; D\Phi \times \tag{3.2}$$
$$\delta\left(\int d^2x_1 d^2x_2 dY\{\frac{\partial\Phi(x_1,x_2;Y)}{\partial Y} + H\Phi(x_1,x_2;Y) + \frac{2\pi\bar{\alpha}_S}{N_c}\int\frac{x_{12}^2 d^2x_3}{x_{23}^2 x_{13}^2}\Phi(x_1,x_3;Y)\Phi(x_3,x_2;Y) + \zeta(x_1,x_2;Y)\}\right)$$

where $P(\zeta)$ stands for Gaussian probability distribution of the noise with $<\zeta(x_1,x_2;Y)>=0$. The noise $\zeta(x_1,x_2;Y)$ has correlation function

$$<\zeta(x_1,x_2;Y)\zeta(x_1',x_2';Y')>= B\;\delta(Y-Y')\;\delta(x_1-x_1')\;\delta(x_2-x_2') \tag{3.3}$$

where B is some function of x_1, x_2 and Y, and will be determined later.

We want to rewrite Eq. (3.2) as a functional integral with a term responsible for merging of Pomerons. To do this we introduce a response field $\tilde{\Phi}(x_1,x_2;Y)$ and use exponential representation of δ-function

$$Z[\Phi,\tilde{\Phi},\zeta] \;\sim \int P(\zeta)\; D\zeta\; D\Phi\; D\tilde{\Phi}\;\; \exp(i\int d^2x_1 d^2x_2 dY\;\tilde{\Phi}(x_1,x_2;Y)\{\frac{\partial\Phi(x_1,x_2;Y)}{\partial Y} + H\Phi(x_1,x_2;Y)$$
$$+\frac{2\pi\bar{\alpha}_S}{N_c}\int\frac{x_{12}^2 d^2x_3}{x_{23}^2 x_{13}^2}\Phi(x_1,x_3;Y)\Phi(x_3,x_2;Y) + \zeta(x_1,x_2;Y)\}) \tag{3.4}$$

Performing functional integration over noise $\zeta(x_1,x_2;Y)$ first, we have

$$Z[\Phi,\tilde{\Phi}] \;\sim \int\; D\Phi\; D\tilde{\Phi}\;\; \exp(i\int d^2x_1 d^2x_2 dY\;\tilde{\Phi}(x_1,x_2;Y)\{\frac{\partial\Phi(x_1,x_2;Y)}{\partial Y} + H\Phi(x_1,x_2;Y)$$
$$+\frac{2\pi\bar{\alpha}_S}{N_c}\int\frac{x_{12}^2 d^2x_3}{x_{23}^2 x_{13}^2}\Phi(x_1,x_3;Y)\Phi(x_3,x_2;Y) + \frac{i}{2}B\;\tilde{\Phi}(x_1,x_2;Y)\}) \tag{3.5}$$

Here we used properties of Gaussian integral. Now we compare functional integral of Eq. (3.4) with one given by Eq. (2.13). It is clear that in order to have proper free and interaction terms we should identify arbitrary $\tilde{\Phi}(x_1,x_2;Y)$ and B as

$$i\;\tilde{\Phi}(x_1,x_2;Y) \equiv p_1^2\;p_2^2\;\Phi^+(x_1,x_2;Y) \tag{3.6}$$

and

$$B \equiv 2\frac{2\pi\bar{\alpha}_S}{N_c}\left(\frac{1}{p_1^2 p_2^2 \Phi^+(x_1,x_2;Y)}\right)^2 \int \frac{d^2x_3}{x_{12}^2 x_{13}^2 x_{23}^2}(L_{12}\Phi(x_1,x_2;Y))\Phi^+(x_1,x_3;Y)\Phi^+(x_3,x_2;Y) \tag{3.7}$$

The last missing thing is to consider functional integration measure. After the redefinition of the response field the measure of functional integration is $D(-i\ p_1^2 p_2^2\ \Phi^+(x_1,x_2;Y))$ instead of desirable $D(\Phi^+(x_1,x_2;Y))$. But according to the definition of the functional integration we have to sum over all possible field configurations. After performing change $D(-i\ p_1^2 p_2^2\ \Phi^+(x_1,x_2;Y)) \to D(\Phi^+(x_1,x_2;Y))$ we still account for all possible configurations up to, maybe, an infinite constant, which cancels in any calculations of physical observables.

Thus, using a Gaussian noise of the form Eq. (3.7) we fully reproduce functional Eq. (2.13), except the terms of Lagrangian corresponding to external sources. As it was already mentioned these terms can be absorbed in the initial condition of Langevin equation. It should be stressed that correspondence between functional and stochastic formulation of the problem was shown on general basis without any additional assumptions.

In the toy model of zero transverse dimensions the action is given by

$$S\ =\ \int dY\left(\Phi^+(Y)\{\frac{d}{dY}-\Delta\}\Phi(Y) + G_{3P}\left(\Phi^+(Y)\Phi^2(Y)+\Phi(Y)\Phi^{+2}(Y)\right) - \Phi^+(Y)\tau_{tr}\delta(Y-Y_0)\right) \tag{3.8}$$

where Δ and G_{3P} are Pomeron intercept and triple Pomeron vertex.

For this form of action Eq. (3.7) becomes

$$B = 2G_{3P}\Phi \tag{3.9}$$

leading to correlation function Eq. (3.3) to be proportional to the field. This means that our theory belongs to direct percolation universality class.

As a final step of our discussion we would like to write Langevin equation used in Eq. (3.2) in a slightly different way. Because of Gaussian form of probability distribution $P(\zeta)$ we may make the following change of variables

$$\zeta(x_1,x_2;Y) \to \sqrt{B}\ \zeta(x_1,x_2;Y) \tag{3.10}$$

In this case corresponding Langevin equation reads

$$\frac{\partial\Phi(x_1,x_2;Y)}{\partial Y} + H\Phi(x_1,x_2;Y) - \frac{2\pi\bar{\alpha}_S}{N_c}\int\frac{x_{12}^2 d^2x_3}{x_{23}^2 x_{13}^2}\Phi(x_1,x_3;Y)\Phi(x_3,x_2;Y) + \sqrt{B}\zeta(x_1,x_2;Y) = 0 \tag{3.11}$$

and correlation function Eq. (3.3) is given by

$$<\zeta(x_1,x_2;Y)\zeta(x_1',x_2';Y')>=\delta(Y-Y')\ \delta(x_1-x_1')\ \delta(x_2-x_2') \tag{3.12}$$

In the toy model the Langevin equation for redefined noise Eq. (3.10) is given by

$$\frac{\partial\Phi(Y)}{\partial Y} = \Delta\Phi(Y) - G_{3P}\Phi^2(Y) - \sqrt{2G_{3P}\Phi(Y)}\zeta(Y) \tag{3.13}$$

Therefore, the description of the BFKL Pomeron calculus with the help of the Langevin equation for directed percolation is equivalent to the functional integral with the action of Eq. (2.13). In QCD we expect (see [20, 21]) a different form of correlation function ($<\zeta\zeta>\propto\Phi(1-\Phi)$) which belongs to universality class different from that of Eq. (3.9).

Going back to coordinate dependence we must admit that the form of noise Eq. (3.7) is too much complicated for calculations, and we will use $L_{1,2}\Phi(1,2)\simeq\Phi(1,2)$ approximation for our further considerations. We believe that this approximation describes well Pomeron dynamics at high energies. The justification of this assumption is given in the next section, where we consider Langevin equation for a mixed momentum and impact parameter representation.

3.2 Langevin equation in mixed representation

At high energies both n and ν are small and one can assume that

$$L_{1,2}\Phi(1,2) \simeq \Phi(1,2) \quad \to \quad \nabla_1^2\nabla_2^2\Phi(1,2) \simeq \frac{1}{x_{12}^4}\Phi(1,2) \tag{3.14}$$

This assumption means that for all BFKL Pomerons in the BFKL Pomeron calculus we can use the diffusion approximation, which is equivalent of using Fisher-Kolmogorov-Petrov-Piskounov equation in statistical approach. In other words, we assume that all partial energies in the BFKL Pomeron calculus are high. In this case the form of the action of Eq. (2.13) simplifies considerably

$$\begin{aligned} S_0 + S_I \,=\, & \int \frac{d^2x_1 d^2x_2}{x_{12}^4}\Phi^+(x_1,x_2;Y)\frac{\partial\Phi(x_1,x_2;Y)}{\partial Y} \\ & + \int dY \frac{d^2x_1 d^2x_2 d^2x_3}{x_{12}^2 x_{23}^2 x_{13}^2}\Big(\frac{\bar\alpha_S}{2\pi}\Phi^+(x_1,x_2;Y)\{\Phi(x_1,x_2;Y)-\Phi(x_1,x_3;Y)-\Phi(x_3,x_2;Y)\} \\ & \qquad + \frac{2\pi\bar\alpha_S^2}{N_c}\{\Phi(x_1,x_2;Y)\Phi^+(x_1,x_3;Y)\Phi^+(x_3,x_2;Y)+h.c.\}\Big) \end{aligned} \tag{3.15}$$

To justify the assumption of Eq. (3.14) we want to show how the simplified action of Eq. (3.15) generates the same BFKL calculus as the full one. To see this we consider the enhanced diagram of Fig. 1-3 using the full action of Eq. (2.13), namely,

$$\begin{aligned} A(Fig.\ 1-3) = & \int \frac{d^2x_1' d^2x_2' d^2x_3'}{x_{12}'^2 x_{13}'^2 x_{23}'^2}\frac{d^2x_1'' d^2x_2'' d^2x_3''}{x_{12}''^2 x_{13}''^2 x_{23}''^2}\int_0^Y dY' \int_0^{Y'} dY'' G(x_1^p,x_2^p;Y|x_1',x_2';Y')\bar{\mathbb{L}}_{1',2'} \\ & \times G(x_1',x_3';Y'|x_1'',x_3'';Y'')G(x_3',x_2';Y'|x_3'',x_2'';Y'')\bar{\mathbb{L}}_{1'',2''}G(x_1'',x_2'';Y''|x_1^t,x_2^t;0) \end{aligned} \tag{3.16}$$

Eq. (2.6) can be written in the following form

$$G(x_1^p,x_2^p;Y|x_1',x_2';Y') = \sum_{n=-\infty}^{\infty}\int d\nu \int d^2x_0 e^{\omega(n,\nu)(Y-Y')}G_n(x_1^p,x_2^p;x_1',x_2'|\nu) \tag{3.17}$$

Therefore

$$\begin{aligned} G(x_1^p,x_2^p;Y|x_1',x_2';Y')\bar{\mathbb{L}}_{1',2'} \quad &= \quad \sum_{n=-\infty}^{\infty}\int d\nu \int d^2x_0 e^{\omega(n,\nu)(Y-Y')}G_n(x_1^p,x_2^p;x_1',x_2'|\nu)\bar{\mathbb{L}}_{1',2'} \\ &= \quad \sum_{n=-\infty}^{\infty}\int d\nu \int d^2x_0 e^{\omega(n,\nu)(Y-Y')}\frac{1}{\lambda(n,\nu)}\,G_n(x_1^p,x_2^p;x_1',x_2'|\nu) \\ &\stackrel{n=0,\nu\to 0}{\longrightarrow} \int d\nu \int d^2x_0 e^{\omega(0,\nu)(Y-Y')}\frac{1}{\lambda(0,0)}\,G_{n=0}(x_1^p,x_2^p;x_1',x_2'|\nu) \\ &\simeq \quad G(x_1^p,x_2^p;Y|x_1',x_2';Y') \end{aligned} \tag{3.18}$$

where we used the fact that $Y-Y'$, $Y'-Y''$ and $Y''-0$ are so large that we can account only for $n=0$ term and replace $\lambda(0,\nu)$ by 1 (see Eq. (2.9)). In Appendix A we show that replacing the sum over n in Eq. (3.17) by the $n=0$ term we obtain the Green function of the BFKL Pomeron that satisfies initial condition determined by the Born approximation. Using Eq. (3.18) we can rewrite Eq. (3.16) as

$$\begin{aligned} A(Fig.\ 1-3) = & \int \frac{d^2x_1' d^2x_2' d^2x_3'}{x_{12}'^2 x_{13}'^2 x_{23}'^2}\frac{d^2x_1'' d^2x_2'' d^2x_3''}{x_{12}''^2 x_{13}''^2 x_{23}''^2}\int_0^Y dY' \int_0^{Y'} dY'' G(x_1^p,x_2^p;Y|x_1',x_2';Y') \\ & \times G(x_1',x_3';Y'|x_1'',x_3'';Y'')G(x_3',x_2';Y'|x_3'',x_2'';Y'')G(x_1'',x_2'';Y''|x_1^t,x_2^t;0) \end{aligned} \tag{3.19}$$

which is the expression for the diagram of Fig. 1-3 for the action in Eq. (3.15).

Eq. (3.15) can be reduced to a more elegant form in mixed momentum and impact parameter representation. We assume that impact parameter is much larger than a dipole size

$$\frac{x_1 + x_2}{2} \gg x_1 - x_2 \tag{3.20}$$

Indeed, we expect that the typical size of the dipoles will be of the order of inverse saturation momentum $1/Q_s(x)$, while the typical impact parameter of the scattering dipole should be much larger (at least of the order of the size of the hadron). Thus, we take $b = (x_1 + x_2)/2 \simeq (x_1 + x_3)/2 \simeq (x_2 + x_3)/2$.

We introduce the Fourier transform

$$\Phi(x_1, x_2; Y) = x_{12}^2 \int d^2k \; e^{i\vec{k}\cdot\vec{x}_{12}} \; \Phi(k, b; Y) \tag{3.21}$$

In terms of fields $\Phi(k, b; Y)$ and $\Phi^+(k, b; Y)$ the action looks as follows

$$S = S_0 + S_I = \int d^2k \; d^2b \; dY \; \Phi^+(k, b; Y) \quad \left(\frac{\partial \Phi(k, b; Y)}{\partial Y} - \frac{\bar{\alpha}_S}{2\pi} \int d^2k' K(k, k') \Phi(k', b; Y) \right.$$
$$\left. + \frac{2\pi \bar{\alpha}_S^2}{N_c} \{ \Phi^+(k, b; Y) \Phi(k, b; Y) + \Phi(k, b; Y) \Phi(k, b; Y) \} \right) \tag{3.22}$$

where $K(k, k')$ is the BFKL kernel in the momentum representation, namely,

$$\int \; d^2 \; k' \; K(k, k') \; \Phi(k', b; Y) \; = \int d^2k' \frac{\Phi(k', b; Y)}{(k - k')^2} - k^2 \int d^2k \; k' \frac{\Phi(k, b; Y)}{(k - k')^2 (k'^2 + (k - k')^2)} \tag{3.23}$$

In Eq. (3.22) we obtained interaction terms multiplied by a constant vertex. In the derivation of Eq. (3.22) we used condition Eq. (3.20), in this limit one can neglect momentum transferred along Pomerons resulting into a constant vertex. The action Eq. (3.23) looks similar to the action of the toy model Eq. (3.8). Following the procedure presented in Section 3 the action of Eq. (3.23) can be easily transformed to Langevin equation for directed percolation

$$\frac{\partial \Phi(k, b; Y)}{\partial Y} = \frac{\bar{\alpha}_S}{2\pi} \int d^2k' K(k, k') \Phi(k', b; Y) - \frac{2\pi \bar{\alpha}_S^2}{N_c} \Phi(k, b; Y) \Phi(k, b; Y) + \zeta(k, b; Y) \tag{3.24}$$

with

$$< \zeta(k, b; Y) >= 0 \tag{3.25}$$

and

$$< \zeta(k, b; Y) \zeta(k', b'; Y) >= \frac{4\pi \bar{\alpha}_S^2}{N_c} \; \Phi(k, b; Y) \; \delta^{(2)}(\vec{b} - \vec{b'}) \; \delta^{(2)}(\vec{k} - \vec{k'}) \; \delta(Y - Y') \tag{3.26}$$

The Langevin equation is the one of many ways to describe a diffusion process and the considerable progress has been achieved in this approach (see [32, 33, 34, 22, 23, 37]).

However, we prefer a different way for description of the BFKL Pomeron interactions, which will also lead to diffusion equation: the so called generating functional approach. The advantage of the generating functional approach is its transparent relation to the partonic wave function of the fast hadron (dipole). In this approach we see in the most explicit way our main theoretical problem: the BFKL Pomeron calculus provides the amplitude that satisfies the t-channel unitarity while the s-channel unitarity is still a problem in the BFKL Pomeron calculus. However, the probabilistic interpretation in the framework of the generating functional leads to the correctly normalized partonic wave function which takes into account the main properties of the s-channel unitarity as well.

3.3 Generating functional: general approach

In this subsection we discuss the main equations of the BFKL Pomeron Calculus in the formalism of the generating functional, which we consider as the most appropriate technique for the probabilistic interpretation of this approach to high energy scattering in QCD.

To begin with let us write down the definition of the generating functional [17]

$$Z\left(Y\,-\,Y_0;\,[u]\right)\;\equiv \tag{3.27}$$

$$\equiv\;\sum_{n=1}\int\;P_n\left(Y\,-\,Y_0;\,x_1,y_1;\ldots;x_i,y_i;\ldots;x_n,y_n\right)\;\prod_{i=1}^{n}u(x_i,y_i)\,d^2\,x_i\,d^2\,y_i$$

where $u(x_i,y_i)\equiv u_i$ is an arbitrary function of x_i and y_i. The coordinates (x_i,y_i) describe the colourless pair of gluons or a dipole. P_n is a probability density to find n dipoles with the size x_i-y_i and impact parameter $(x_i+y_i)/2$. It follows directly from the physical meaning of P_n and the definition in Eq. (3.27) that the functional obeys the condition

$$Z\left(Y\,-\,Y_0;\,[u=1]\right)\;=\;1\,. \tag{3.28}$$

The physical meaning of this equation is that the sum over all probabilities is equal to unity.

Introducing vertices for the dipole process: $1\to 2$ ($V_{1\to2}(x,y\to x_1,y_1+x_2,y_2)$) and $2\to 1$ ($V_{2\to1}(x_1,y_1+x_2,y_2\to x,y)$) we can write a typical birth-death equation in the form

$$\frac{\partial\,P_n(Y;\ldots;x_i,y_i;\ldots;x_n,y_n)}{\partial Y}\;\;= \tag{3.29}$$

$$=\;\sum_i\;V_{1\to2}\bigotimes\left(P_{n-1}(Y;\ldots;x_i,y_i;\ldots;x_n,y_n)\;-\;P_n(Y;\ldots;x_i,y_i;\ldots;x_n,y_n)\right) \tag{3.30}$$

$$+\;\sum_{i>k}V_{2\to1}\bigotimes\left(P_{n+1}(Y;\ldots;x_i,y_i;\ldots;x_k,y_k;\ldots;x_n,y_n)\;-\;P_n(Y;\ldots;x_i,y_i;\ldots;x_k,y_k;\ldots;x_n,y_n)\right) \tag{3.31}$$

Eq. (3.29) is the typical Markov chain and the fact that we have the correct normalized partonic wave function is written in Eq. (3.29) by introducing for each microscopic (dipole) process two terms (see Eq. (3.30) and Eq. (3.31)): the emission of dipoles (positive birth term) and their recombination (negative death term). Multiplying this equation by the product $\prod_{i=1}^{n}u_i$ and integrating over all x_i and y_i, we obtain the following linear equation for the generating functional:

$$\frac{\partial\,Z\;(Y-Y_0;[\,u\,])}{\partial\,Y}\;\;=\;\chi\,[\,u\,]\;Z\;(Y-Y_0;[\,u\,]) \tag{3.32}$$

with

$$\chi[u]\;\;=\;\;\int\,d^4\,qd^4q_1\,d^4q_2\;\left(V_{1\,\to\,2}\,(q\to q_1+q_2)\;(-u(q)\;+\;u(q_1)\,u(q_2)\,)\;\frac{\delta}{\delta u(q)}\right. \tag{3.33}$$

$$\left.-V_{2\,\to\,1}\,(q_1+q_2\to q)\;(u(q_1)\,u(q_2)\;-\;u(q))\;\frac{1}{2}\,\frac{\delta^2}{\delta u(q_1)\,\delta u(q_2)}\right) \tag{3.34}$$

These equations describe the process of splitting of one Pomeron into two Pomerons and merging of two Pomerons into one Pomeron. We will prove below, that this set of equations is equivalent to the BFKL Pomeron calculus, given by the path integral of Eq. (2.13). We use notations q_i for (x_i,y_i) and d^4q_i for $d^2x_i\,d^2\,y_i$ where x_i and y_i are positions of quark (antiquark) of the colourless dipole.

Eq. (3.32) is a typical diffusion equation or Fokker-Planck equation [13], with the diffusion coefficient which depends on u. This is the master equation of our approach, and the goal of this paper is to find the correspondence

between this equation and the BFKL Pomeron Calculus. In spite of the fact that this is a functional equation we intuitively feel that this equation could be useful since we can develop a direct method for its solution and, on the other hand, there exist many studies of such an equation in the literature (see for example [13]) as well as some physical realizations in statistical physics. The intimate relation between the Fokker-Planck equation, and the high energy asymptotic was first pointed out by Weigert [33] in JIMWLK approach [38], and has been discussed in [34, 22, 23].

The scattering amplitude can be defined as a functional [29, 19]

$$\begin{aligned} N\left(Y;[\gamma_i]\right) &= -\sum_{n=1}^{\infty}\int (-1)^n \gamma_n(x_1,y_1;\ldots;x_n,y_n;Y_0)\prod_{i=1}^{n}\frac{\delta}{\delta u_i} Z\left(Y,[u_i]\right)|_{u_i=1}\, d^2x_i\, d^2y_i \\ &= -\sum_{n=1}^{\infty}(-1)^n\int \gamma_n(x_1,y_1;\ldots;x_n,y_n;Y_0)\;\rho(x_1,y_1;\ldots;x_n,y_n;Y-Y_0)\, d^2x_i\, d^2y_i \end{aligned} \tag{3.35}$$

The physical meaning of functions γ_n is the imaginary part of the amplitude of interaction of n-dipoles with the target at low energies. All these functions should be taken from the non-pertubative QCD input. However, in [18, 19, 20] it was shown that we can introduce the amplitude of interaction of n-dipoles $\gamma_n(x_1,y_1;\ldots;x_n,y_n;Y)$ at high energies (large values of rapidity Y) and Eq. (3.27), Eq. (3.32) and Eq. (3.35) can be rewritten as a chain set of equation for $\gamma_n(x_1,y_1;\ldots;x_n,y_n;Y)$. The equation has the form[5]

$$\begin{aligned} \frac{\partial\,\gamma_n\left(q_1\ldots,q_n\right)}{\partial\, Y} &= 2\sum_{i=1}^{n}\int d^4q'\, d^4q\, V_{1\to 2}(q_i;\,q,\,q')\,\gamma_n\left(\ldots q'\ldots\right) - \sum_{i=1}^{n}\int d^4q_1'\, d^4q_2'\, V_{1\to 2}(q_i;\,q_1',\,q_2')\,\gamma_n\left(\ldots,q_i\ldots\right) \\ &- \sum_{i=1}^{n-1}\int d^4q\, d^4q' V_{1\to 2}(q_i;\,q,\,q')\,\gamma_{n+1}\left(\ldots q\ldots q'\right) - \sum_{i>j}^{n}\int d^4\,q\, V_{2\to 1}\left(q_i,\,q_j;\,q\right)\gamma_{n-1}\left(q_i\ldots q_j\ldots q\right) \\ &+\; 2\sum_{i=1}^{n}\int d^4q\, d^4q' V_{2\to 1}\left(q,q_i;q'\right)\;\gamma_{n-1}\left(\ldots q_i\ldots q\right) + \sum_{i>k}^{n}\int d^4q\, V_{2\to 1}\left(q_i,q_k;q\right)\;\gamma_n\left(\ldots q_i\ldots q_k\ldots\right) \end{aligned} \tag{3.36}$$

Comparing this equation for $\gamma_1 \equiv T^{(1)}$ and $\gamma_2 \equiv T^{(2)}$ one can see that

$$V_{1\to 2} = \frac{\bar{\alpha}_S}{2\,\pi}\Gamma_{1\to 2} = \frac{\bar{\alpha}_S}{2\,\pi}\,K\left(x,y;z\right)\,; \tag{3.37}$$

$$\begin{aligned} V_{2\to 1} &= \frac{2\,\alpha_S^2}{\pi^2}\frac{\bar{\alpha}_S}{2\,\pi}\left(-\frac{\Gamma_{2\to 1}(x_1,y_1+x_2,y_2\to x,y)}{(x-y)^4}+\right. \\ &+\left.\int\frac{d^2x\,d^2y}{(x-y)^4}\,\Gamma_{2\to 1}(x_1,y_1+x_2,y_2\to x,y)\,\left(\delta^{(2)}(x_1-x)\delta^{(2)}(y_1-y)\;+\;\delta^{(2)}(x_2-x)\delta^{(2)}(y_2-y)\right)\right) \end{aligned} \tag{3.38}$$

with $\Gamma_{2\to 1}$ is given by Eq. (2.35).

3.4 A toy model: Pomeron interaction and probabilistic interpretation

In this section we consider the simple toy model in which the probabilities to find n-dipoles being independent on dipoles sizes [17, 18, 20, 21]. In this model the master equation (3.32) has a simple form

$$\frac{\partial Z}{\partial Y} = -\Gamma(1\to 2)\,u(1-u)\,\frac{\partial Z}{\partial u} + \frac{1}{2}\,\Gamma(2\to 1)\,u(1-u)\,\frac{\partial^2 Z}{\partial u^2} \tag{3.39}$$

[5]This equation is Eq. (2.19) in [20] but, hopefully, without misprints, part of which has been noticed in [27].

Eq. (3.39) generates: the Pomeron splitting $G_{P\to 2P} = \Gamma(1 \to 2)$; Pomerons merging $G_{2P\to P} = \frac{1}{2}\Gamma(2 \to 1)$ and also the two Pomerons scattering $G_{2P\to 2P} = \frac{1}{2}\Gamma(2 \to 1)$. It is easy to see that by neglecting $u^2\partial^2 Z/\partial u^2$ term in Eq. (3.39) we cannot provide a correct sign for Pomerons merging $G_{2P\to P}$. It worth mentioning that we need this term to provide the conservation of the total probability on each level of rapidity.

The description given by Eq. (3.39) is equivalent to the path integral of Eq. (3.8). To see this we need to notice that the general solution of Eq. (3.39) has a form

$$Z(Y;u) \;=\; e^{H(u)\,(Y-Y_0)}\,Z(Y_0;u) \tag{3.40}$$

with operator H defined as

$$H(u) \;=\; -\,\Gamma(1\to 2)\,u(1-u)\,\frac{\partial}{\partial u} + \frac{1}{2}\Gamma(2\to 1)\,u(1-u)\,\frac{\partial^2}{\partial u^2} \tag{3.41}$$

and

$$Z(Y_0;u) \;=\; e^{\tau_{tr}(u-1)} \tag{3.42}$$

Introducing operators of creation (a^+) and annihilation (a)

$$\hat{a} \,=\, \frac{\partial}{\partial u}\,; \qquad \hat{a}^+ \,=\, u \;\text{ that satisfy } [\hat{a},\hat{a}^+] = 1 \;\text{ at fixed } Y \tag{3.43}$$

one can see that operator $\mathcal{H}$ has the form

$$\mathcal{H} \;=\; -\,\Gamma(1\to 2)\,\hat{a}^+\,(1-\hat{a}^+)\,\hat{a} \;+\; \frac{1}{2}\Gamma(2\to 1)\,\hat{a}^+\,(1-\hat{a}^+)\,\hat{a}^2 \tag{3.44}$$

and the initial state at $Y = Y_0$ is defined as

$$|Y_0> = \; e^{\tau_{tr}(\hat{a}^+ - 1)}|0> \tag{3.45}$$

with the vacuum defined by $\hat{a}|0>=0$.

We need to discretize the development operator of Eq. (3.40) with $\mathcal{H}$ given by Eq. (3.44), namely,

$$e^{\mathcal{H}\,(Y-Y_0)} = e^{\mathcal{H}\,\Delta Y}\;\ldots\;e^{\mathcal{H}\,\Delta Y} = \prod_{j=1}^{N}\,(1\,+\,\mathcal{H}\,\Delta Y) \tag{3.46}$$

and introduce coherent states [39] for a certain intermediate rapidity $Y_j = Y_0 + j\Delta Y$ as

$$|\phi_j> = \; e^{\phi_j\,\hat{a}^+ \,-\,\phi_j}\,|0> \tag{3.47}$$

where ϕ_j are arbitrary complex numbers. The initial state of Eq. (3.45) can be written as

$$|\phi_0(Y_0)> \equiv |\tau_{tr}>\,. \tag{3.48}$$

The unit operator in terms of the coherent states can be expressed as

$$1 \;=\; \left(\int \frac{d\phi_j d\phi_j^*}{\pi i}\,e^{-\phi_j\,\phi_j^* + \phi_j + \phi_j^*}\right)|\phi_j><\phi_j| \tag{3.49}$$

We want to calculate matrix element of some operator A between states of initial Y_0 and final Y rapidity $<Y|A|Y_0>$. This can be written as

$$< Y|A|Y_0 >=< Y|A\left\{\left(\int \frac{d\phi_Y d\phi_Y^*}{\pi i}\, e^{-\phi_Y\,\phi_Y^* + \phi_Y + \phi_Y^*}\right)|\phi_Y >< \phi_Y|\right\}|Y_0 > \tag{3.50}$$

here we denote $|Y>\equiv|\phi_Y>$. Next we use the development operator given in Eq. (3.46) to find $<Y|Y_0>$. We split the rapidity $Y - Y_0$ to N intervals and insert the development Eq. (3.46) and unit Eq. (3.49) operator between the states of intermediate rapidity

$$< Y|\prod_{j=1}^{N}\,(1\,+\,\mathcal{H}\,\Delta Y)\,|Y_0> \tag{3.51}$$

We look at

$$\begin{aligned}
< \phi_{j+1}|\,(1\,+\,\mathcal{H}\,\Delta Y)\,|\phi_j > &= \exp\left\{-\phi_{j+1}^*\phi_{j+1} + \phi_{j+1}^* + \phi_{j+1} - \phi_{j+1}^* - \phi_j + \phi_{j+1}^*\phi_j\right\} \\
&\times\,\left(1\,+\,\mathcal{H}(\phi_{j+1}^*,\phi_j)\,\Delta Y\right) \\
&= \exp\left\{-\phi_{j+1}^*(\phi_{j+1}-\phi_j) + \phi_{j+1} - \phi_j\right\}\left(1\,+\,\mathcal{H}(\phi_{j+1}^*,\phi_j)\,\Delta Y\right) \\
&= \exp\left\{-\phi_{j+1}^*(\phi_{j+1}-\phi_j) + \phi_{j+1} - \phi_j\right\}\,\exp\left(\mathcal{H}(\phi_{j+1}^*,\phi_j)\,\Delta Y\right)
\end{aligned} \tag{3.52}$$

Now we redefine an arbitrary function ϕ_j as

$$\Phi_j^+ \;=\; -\phi_j, \qquad\qquad \Phi_j = 1 - \phi_j^* \tag{3.53}$$

and rewrite Eq. (3.52) in terms of Φ_j and Φ_j^+

$$\begin{aligned}
& e^{-\phi_{j+1}^*(\phi_{j+1}-\phi_j)+\phi_{j+1}-\phi_j}\;\exp\left\{\mathcal{H}(\phi_{j+1}^*,\phi_j)\,\Delta Y\right\}\;= \\
&= \exp\left\{\Phi_{j+1}(\Phi_{j+1}^+ - \Phi_j^+) + \mathcal{H}(1-\Phi_{j+1},-\Phi_j^+)\right\} \\
&= \exp\left\{\left(\frac{\Phi_{j+1}(\Phi_{j+1}^+ - \Phi_j^+)}{\Delta Y} + \mathcal{H}(1-\Phi_{j+1},-\Phi_j^+)\right)\Delta Y\right\}
\end{aligned} \tag{3.54}$$

Summing over all rapidity intervals we have

$$< Y|A|Y_0 > \;\sim\; \prod_{j=0}^{N}\int d\Phi_j^+ d\Phi_j A(Y) e^S \tag{3.55}$$

where $A(Y)$ is the expectation value of the operator A at the final rapidity Y, and

$$S \;=\; \left(\frac{\Phi_{j+1}(\Phi_{j+1}^+ - \Phi_j^+)}{\Delta Y} + \mathcal{H}(1-\Phi_{j+1},-\Phi_j^+)\right)\Delta Y \tag{3.56}$$

In the continuous limit this becomes

$$< Y|A|Y_0 > \;=\; \frac{\int \mathcal{D}\Phi^+\mathcal{D}\Phi\;A(Y)\;e^S}{\int \mathcal{D}\Phi^+\mathcal{D}\Phi\;e^S} \tag{3.57}$$

with

$$\begin{aligned}
S \;&=\; \int dY\left(\Phi^+\frac{d}{dY}\Phi + \mathcal{H}(\Phi+1,-\Phi^+)\right) \\
&=\; \int dY\left(\Phi^+\frac{d}{dY}\Phi\;-\Gamma(1\to 2)\Phi^+\Phi + \Gamma(1\to 2)\Phi^+\Phi^2 + \frac{1}{2}\Gamma(2\to 1)\Phi^{+2}\Phi - \frac{1}{2}\Gamma(2\to 1)\Phi^{+2}\,\Phi^2\right)
\end{aligned} \tag{3.58}$$

Eq. (3.58) is the action of Eq. (3.8) for $\Gamma(1\to 2) = \frac{1}{2}\Gamma(2\to 1)$. In the toy-model the difference between these two vertices is the normalization problem of functions Φ^+ and Φ. In our approach they are normalized in the way

which allows us to treat them as probabilities (see Eq. (2.17) and Eq. (2.18)). However, Eq. (3.58) includes the new interaction: a transition of two Pomerons to two Pomerons. The sign is such that this interaction provides the stability of the potential energy. Indeed this term is responsible for the increase of the potential energy at large values of both Φ^+ and Φ.

Comparing Eq. (3.58) with Eq. (3.1) one can see that we built the partition function and the thermodynamic potential using the generating functional. It means that our Eq. (3.32) is equivalent to statistical description of the system of dipoles.

Eq. (3.39) is the diffusion with the u dependence in diffusion coefficient. In terms of the Langevin equation Eq. (3.58) generates a noise term of the $< \zeta\zeta > \propto \Phi(1-\Phi)$ type.

To out taste Eq. (3.39) is simpler than the Langevin equation of Eq. (3.13) and it will be easily generalized for the case of QCD. For $u < 1$ the diffusion coefficient is positive and the equation has a reasonable solution. If $u > 1$, the sign of this coefficient changes and the equation gives a solution which increases with Y and $Z(Y)$ cannot be treated as the generating function for the probabilities to find n dipoles (Pomerons) (see [10, 12, 21] for details). The same features one can notice in the asymptotic solution that is the solution to Eq. (3.39) with the l.h.c. equal to zero. It is easy to see that this solution has the form

$$Z(Y \to \infty; u) \;=\; \frac{1 - e^{\kappa u}}{1 - e^{\kappa}}\,; \;\; \text{with } \kappa \,=\, 2\frac{\Gamma(1\to 2)}{\Gamma(2\to 1)} = 2\,\frac{2\,N_c^2}{\bar{\alpha}_S^2} \,\gg\, 1 \tag{3.59}$$

For negative κ this solution leads to $Z > 1$ for $u < 1$. This shows that we cannot give a probabilistic interpretation for such a solution.

4. A practical way to find solution: Monte Carlo simulation

In this section we consider the BFKL Pomeron calculus in the form of Eq. (3.22) which leads to the simplest approach in the framework of the generating functional technique. This approach is not only the simplest one but also it is free from all troubles related to the negative contribution for the process of $2P \to P$ transition. Repeating procedure discussed in section 3.2 one can see that Eq. (3.22) for action leads to the following equation for the generating functional

$$\frac{\partial\, Z\;(Y - Y_0; [\,u(b,k)\,])}{\partial\, Y} \;=\; \chi\,[\,u(b,k)\,]\;Z\;(Y - Y_0; [\,u(b,k)\,]) \tag{4.1}$$

with

$$\chi[u] \;=\; \int d^2\,b\,d^2\,k\,\left(\frac{\bar{\alpha}_S}{2\pi}\left(-\int d^2\,k'\,K(k,k')u(b,k')\,\frac{\delta}{\delta u(b,k')} \,+\, u(b,k)\,u(b,k)\,\frac{\delta}{\delta u(b,k)}\right)\right. \tag{4.2}$$

$$\left. -\left(\frac{4\,\pi^2\bar{\alpha}_S}{N_c}\right)^2\,\frac{\bar{\alpha}_S}{2\pi}\,(u(b,k)\,u(b,k) \,-\, u(b,k))\;\frac{1}{2}\,\frac{\delta^2}{\delta u(b,k)\,\delta u(b,k)}\right) \tag{4.3}$$

For $Z(Y,[u(b,k)])$ defined as

$$Z\,(Y \,-\, Y_0;\,[u(b_i,k_i)]) \;\equiv \tag{4.4}$$

$$\equiv\; \sum_{n=1}\int\; P_n\,(Y \,-\, Y_0;\,b_1,k_1;\ldots;b_i,k_i;\ldots;b_n,k_n)\;\prod_{i=1}^{n} u(b_i,k_i)\,d^2\,x_i\,d^2\,k_i$$

where $u(b_i,k_i)$ are arbitrary functions.

Eq. (4.2) and Eq. (4.3) show that the $2P \to 1P$ transition can be written as the two dipole to one dipole merging with a positive probability. Therefore, Eq. (4.1) has a very simple probabilistic interpretation which can be written as the following Markov chain:

$$\frac{\partial\, P_n(Y;\ldots;b_i = b,k_i = k;\ldots;b_n = b,k_n = k)}{\partial Y} \;= \tag{4.5}$$

$$= \frac{\bar{\alpha}_S}{2\pi}\sum_i \left(P_{n-1}(Y;\ldots;b_i=b,k_i=k;\ldots;b_{n-1},k_{n-1}) - \right.$$
$$\left. - \int d^2\,k'\,K(k,k')\;P_n(Y;\ldots;b_i=b,k_i=k';\ldots;x_n=b,k_n=k)\right) \quad (4.6)$$
$$+ \left(\frac{4\,\pi^2\bar{\alpha}_S}{N_c}\right)^2 \frac{\bar{\alpha}_S}{2\pi}\sum_{i>j}\left(P_{n+1}(Y;\ldots;b_j=b,k_j=k;\ldots;b_i=b,k_i=k;\ldots;b_n=b,k_n=k) - \right.$$
$$\left. - P_n(Y;\ldots;b_j=b,k_j=k;\ldots b_i=b,k_i=k;\ldots;b_n=b,k_n=k;)\right); \quad (4.7)$$

This set of equations can be solved numerically and it gives a practical way to discuss the influence of the Pomeron loops on the solution for the scattering amplitude at high energies.

5. Conclusions

We demonstrate in this paper that the BFKL Pomeron Calculus in the kinematic region given by Eq. (1.3) has two equivalent descriptions: (i) one is the generating functional which gives a clear probabilistic interpretation of the processes of high energy scattering and provides also a Hamiltonian-like description of the system of interacting dipoles; (ii) the second is the Langevin equation with a noise term of Eq. (3.7) which is rather complicated. We show that at high energies the Langevin equation with noise of Eq. (3.7) can be reduced to the Langevin equation for directed percolation in the momentum representation if the impact parameter is large, namely, $b \gg 1/k$ where k is the transverse momentum of a dipole. Unfortunately, this simplified form of Langevin equation is not applicable for summation of Pomeron loops, where one integrates over all possible values of impact parameter.

In other words, the BFKL Pomeron Calculus can be considered as an alternative description of the statistical system of dipoles with different kinds of interactions between them. The same conclusion was drawn in [55] using a similar technique. We show that the BFKL Pomeron calculus with two vertices: splitting of one Pomeron into two Pomerons($P \to 2\,P$) and merging of two Pomerons into one Pomeron ($2P \to P$), can be described as a system of colourless dipoles with two processes: the decay of one dipole into two dipoles and the merging of two dipoles into one dipole.

It is shown that the question about negative amplitude does not arise if we treat the system of dipoles in the momentum representation. Markov chain for this system is written in the paper (see Eq. (4.5)) and can be considered as a practical way to find a solution in accessible range of energies. However, as we have stressed a number of assumptions we have made on the way to simplify the noise term as well as to argue the probabilistic interpretation of the Pomeron calculus. Therefore, both these approaches can be considered as a QCD motivated models. In the case of the probabilistic interpretation we can even expect that this approach will not work in the form, that has been suggested in this paper, for the full ($2P \to P$) vertex (see [27]).

Being elegant and beautiful the BFKL Pomeron Calculus has a clear disadvantage: it lacks theoretical ideas what kind of Pomeron interactions we should take into account and why. Of course, Feynman diagrams in leading $\ln(1/x)$ approximation of perturbative QCD allow us, in principle, to calculate all possible Pomeron interactions but, practically, it is very hard job. Even if we calculate these vertices we need to understand what set of vertices we should take into account for the calculation of the scattering amplitude. This is the reason why we need to develop a more general formalism. Fortunately, such a formalism has been built and it is known under the abbreviation JIMWLK-Balitsky approach [41, 38, 28]. In this approach we are able to calculate all vertices for Pomeron interactions as it was demonstrated in [46] and it solves the first part of the problem: determination of all possible Pomeron interactions. However, we need to understand what vertices we should take into account for calculation of the scattering amplitude. We hope that a further progress in going beyond of the BFKL Pomeron Calculus (see [46, 47]) will lead to such a development of the BFKL Pomeron Calculus with a consistent theoretical approach. Hopefully this approach will be simpler than Lipatov effective action [48] which is not easier to solve than the full QCD Lagrangian.

It is well known that the mean field approach to our problem, which includes only one dipole to two dipoles decay in the master equation (see Eq. (3.32)) has been studied quite well both analytically [51] and numerically [53]. We firmly believe that the probabilistic interpretation in spite of being a QCD motivated model, gives a practical method for creating a Monte Carlo code in spirit of the approach suggested in [54]. This code will allow us to find a numerical solution to the problem and to consider inclusive observables. This extension is very desirable since the most experimental data exist for these observables.

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A. Calculation of $G_0(x_1, x_2|x_1', x_2')$

The solution of the BFKL equation is given by [26]

$$G(x_1,x_2;x_1^{'},x_2^{'}|\omega) \;=\; \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{(\nu^2+n^2/4)d\nu}{[\nu^2+(n-1)^2/4][\nu^2+(n+1)^2/4]}\frac{G_{\nu\mu}(x_1,x_2,x_1^{'},x_2^{'})}{\omega-\omega(\nu,\mu)} \tag{A.1}$$

where $G_{\nu\mu}(x_1,x_2,x_1^{'},x_2^{'})$ is the Mellin transform of Eq. (2.6) and x_i are two-dimensional vectors in complex coordinates

$$x_i = x_{i,x} + ix_{i,y} \quad x_i^* = x_{i,x} - ix_{i,y} \tag{A.2}$$

The function $\omega(\nu,\mu)$ is the eigen value of the BFKL equation given by Eq. (2.8).

The four-point Green function is presented in terms of the hypergeometric functions [26, 6]

$$\begin{aligned} G_{\nu\mu}(x_1,x_2,x_1^{'},x_2^{'}) = C_1 x^h x^{*\tilde{h}} F(h,h,2h;x)F(\tilde{h},\tilde{h},2\tilde{h};x^*) \\ +C_2 x^{1-h}x^{*1-\tilde{h}}F(1-h,1-h,2-2h;x)F(1-\tilde{h},1-\tilde{h},2-2\tilde{h};x^*) \end{aligned} \tag{A.3}$$

with $h=\frac{1}{2}+i\nu+\frac{n}{2}$, $\tilde{h}=\frac{1}{2}+i\nu-\frac{n}{2}$, and x is the anharmonic ratio

$$x = \frac{x_{12}x_{1'2'}}{x_{11'}x_{22'}} \tag{A.4}$$

Coefficients C_1 and C_2 are given by [26]

$$C_1 = \frac{b_{n,-\nu}}{2\pi^2} \qquad\qquad C_2 = \frac{b_{n,\nu}}{2\pi^2} \tag{A.5}$$

with

$$b_{n,\nu} = \pi^3 2^{4i\nu}\frac{\Gamma(-i\nu+(1+|n|)/2)\Gamma(i\nu+|n|/2)}{\Gamma(i\nu+(1+|n|)/2)\Gamma(-i\nu+|n|/2)} \tag{A.6}$$

As we have discussed, the high energy asymptotic behaviour stems from $n = 0$ term in Eq. (A.1). The initial condition for Eq. (2.6) at $Y = Y_0$ is given by the following expression

$$G_0(x_1, x_2; x_1', x_2') \; = \tag{A.7}$$

$$\int \frac{d\omega}{2\,\pi\,i} \int_{-\infty}^{+\infty} \frac{\nu^2 d\nu}{(\nu^2+1/4)^2} \frac{G_{\nu\mu}(x_1, x_2, x_1', x_2')}{\omega - \omega(\nu, n=0)} \; = \; \frac{1}{\omega} \int_{-\infty}^{+\infty} \frac{\nu^2 d\nu}{(\nu^2+1/4)^2} \; G_{\nu\mu}(x_1, x_2, x_1', x_2')$$

This integral can be taken by closing contour of integration over singularities of the integrand.

The function

$$\frac{\nu^2}{(\nu^2+1/4)^2} \tag{A.8}$$

has two poles at $\frac{i}{2}$ and $-\frac{i}{2}$. The four-point Green function of Eq. (A.3) consists of two terms, one with $C_1 (xx^*)^{\frac{1}{2}+i\nu}$ and the other one $C_2 (xx^*)^{\frac{1}{2}-i\nu}$. For small $|x|$ this terms should be integrated closing contour in upper and lower semi-planes respectively. The resulting contour in the lower semi-plane runs anticlockwise and thus the value of the contour integral enters with a minus sign.

The terms could be expanded in the vicinity of their poles. Let us consider the first term. We expand the function C_1 in the vicinity of $\frac{i}{2}$

$$C_1 = \frac{\pi}{2} 2^{-4i\nu} \frac{\Gamma(-i\nu)}{\Gamma(-i\nu+\frac{1}{2})\Gamma(1+i\nu)} \frac{\Gamma(i\nu+\frac{3}{2})}{(i\nu+\frac{1}{2})} \tag{A.9}$$

The hypergeometric function can be written as a sum

$$F(a,b,c;x) = 1 + \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=1}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!} \tag{A.10}$$

In the case of $F(h,h,2h;x)$ the singularity of Γ function at $\frac{i}{2}$ can factorized out the sum

$$\begin{aligned} F(h,h,2h;x) &= 1 + \frac{\Gamma(2h)}{\Gamma(h)\Gamma(h)} \sum_{n=1}^{\infty} \frac{\Gamma(h+n)\Gamma(h+n)}{\Gamma(2h+n)} \frac{x^n}{n!} \\ &\simeq 1 + \frac{1}{2\Gamma(h)} \sum_{n=1}^{\infty} \frac{x^n}{n} = 1 - \frac{1}{2} \frac{ln(1-x)}{\Gamma(i\nu+\frac{1}{2})} \end{aligned} \tag{A.11}$$

At this stage the first term of the integrand of Eq. (A.7) can be written as

$$\frac{1}{\omega} \frac{\nu^2}{(i\nu+\frac{1}{2})^3 (i\nu-\frac{1}{2})^2} \frac{\pi}{2} 2^{-4i\nu} \frac{\Gamma(-i\nu)}{\Gamma(-i\nu+\frac{1}{2})\Gamma(1+i\nu)} (xx^*)^{\frac{1}{2}+i\nu} \left(1 - \frac{1}{2} \frac{ln(1-x)}{\Gamma(\frac{1}{2}+i\nu)}\right) \left(1 - \frac{1}{2} \frac{ln(1-x^*)}{\Gamma(\frac{1}{2}+i\nu)}\right) \tag{A.12}$$

It is clearly seen that the term of zero order in $ln|1-x|$ has a third order pole at $\nu = \frac{i}{2}$; the term of first order in $ln|1-x|$ has a second order pole at $\nu = \frac{i}{2}$, and, the term of second order in $ln|1-x|$ has a simple pole at $\nu = \frac{i}{2}$. The contributions of those terms in the contour integral are found to be

$$\frac{i\pi^2}{\omega} [-2 \; + \; ln(xx^*)(-4 + ln(xx^*))] \tag{A.13}$$

$$-\frac{\pi^2}{2\omega} [-2 \; + \; \gamma \; + \; ln(xx^*)] ln[(1-x)(1-x^*)] \tag{A.14}$$

$$-\frac{i\pi^2}{4\omega} ln(1-x) ln(1-x^*) \tag{A.15}$$

respectively.

In a similar way we may expand the second of the integrand in vicinity of its pole at $\nu = -\frac{i}{2}$, namely

$$\frac{1}{\omega}\frac{\nu^2}{(\frac{1}{2}-i\nu)^3(i\nu+\frac{1}{2})^2}\frac{\pi}{2}2^{4i\nu}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\frac{1}{2})\Gamma(1-i\nu)}(xx^*)^{\frac{1}{2}-i\nu}\left(1-\frac{1}{2}\frac{ln(1-x)}{\Gamma(\frac{1}{2}-i\nu)}\right)\left(1-\frac{1}{2}\frac{ln(1-x^*)}{\Gamma(\frac{1}{2}-i\nu)}\right) \tag{A.16}$$

The integration is performed on the lower semicircle and results in overall minus sign of the integral. The contributions corresponding to Eq. (A.13), Eq. (A.14) and Eq. (A.15) are

$$-\frac{i\pi^2}{\omega}[-\ 2\ +\ ln(xx^*)(-4+ln(xx^*))] \tag{A.17}$$

$$-\frac{\pi^2}{2\omega}[-2\ +\ \gamma\ +\ ln(xx^*)]ln[(1-x)(1-x^*)] \tag{A.18}$$

$$+\frac{i\pi^2}{4\omega} ln(1-x) ln(1-x^*) \tag{A.19}$$

respectively.

Comparing the contributions we note that that of zero and second order in $ln[(1-x)(1-x^*)]$ are exactly canceled out, and we are left with

$$-2\frac{\pi^2}{2\omega}[-2\ +\ \gamma\ +\ ln(xx^*)]ln[(1-x)(1-x^*)] \tag{A.20}$$

For small $|x|$ this can written as

$$-\frac{4\pi^2}{\omega}\ ln|x|\ ln|1-x| \tag{A.21}$$

or

$$+\frac{4\pi^2}{\omega}\ ln\frac{1}{|x|}\ ln|1-x| \tag{A.22}$$

Going back to complex vector representation of x and rewriting Eq. (A.23) as

$$G_0(x_1,x_2;x_1',x_2') \ =\ +\frac{4\pi^2}{\omega}\ ln\left|\frac{x_{11'}x_{22'}}{x_{12'}x_{1'2}}\right|\ ln\left|\frac{x_{11'}x_{22'}}{x_{12}x_{1'2'}}\right| \tag{A.23}$$

we see that we reproduce the result of [26]. Therefore, we demonstrated that Eq. (A.23) gives a correct initial condition for searching the scattering amplitude at high energies restricting ourselves by the one term in Eq. (2.6) with $n = 0$.

B. The path integral formalism for the generating functional

We want to develop a path integral formalism similar to that we found for the toy model, but where the probabilities to find n dipoles depend of dipole sizes. In our notation we denote by Latin index rapidity interval, and Greek indices relate to a size of dipole.

As in Chapter 3.3 we introduce the creation and annihilation operators

$$\hat{a}(q) = \frac{\delta}{\delta u(q)} \qquad\qquad \hat{a}^\dagger(q) = u(q) \tag{B.1}$$

with commutation relations $[\hat{a}(q), \hat{a}^\dagger(q')] = \delta(q - q')$ at fixed Y. The expression for the coherent states in this case takes form of

$$|\phi(q)>= e^{\phi(q)\hat{a}^\dagger(q) - \phi(q)}|0> \tag{B.2}$$

with

$$\hat{a}(q')|\phi(q)>= \phi(q)|\phi(q)> \delta(q - q') \tag{B.3}$$

First we consider discrete dipole sizes $q_\alpha = (L/N)\alpha$, where L is a maximal possible dipole size, N a number of intervals of L discretization, and α is an integer number running from 0 to N. In this case the commutation relations become $[\hat{a}(q_\alpha), \hat{a}^\dagger(q_\beta)] = \delta_{\alpha,\beta}$.

The unit operator can be written in terms of the coherent states

$$\hat{I} = \prod_\alpha \int \frac{d\phi^*(q_\alpha) d\phi(q_\alpha)}{i\pi} e^{-\phi^*(q_\alpha)\phi(q_\alpha) + \phi(q_\alpha) + \phi^*(q_\alpha)} |\phi(q_\alpha)>< \phi(q_\alpha)| \tag{B.4}$$

The operator $\mathcal{H}$ defined in Eq. (3.33) can be written as

$$\begin{aligned} \mathcal{H}[\hat{a}^\dagger, \hat{a}] = & - \sum_\beta \sum_\gamma \sum_\lambda [V_{1\to 2}(q_\beta \to q_\gamma + q_\lambda)\{-\hat{a}^\dagger(q_\beta) + \hat{a}^\dagger(q_\gamma)\hat{a}^\dagger(q_\lambda)\}\hat{a}(q_\beta) \\ & - V_{2\to 1}(q_\gamma + q_\lambda \to q_\beta)\{\hat{a}^\dagger(q_\gamma)\hat{a}^\dagger(q_\lambda) - \hat{a}^\dagger(q_\beta)\}\frac{1}{2}\hat{a}(q_\gamma)\hat{a}(q_\lambda)] \end{aligned} \tag{B.5}$$

Following the logic of Chapter 3.3 we consider a matrix element

$$\left\{\prod_{\alpha'} < \phi_{j+1}(q_{\alpha'})|\right\} (1 + \mathcal{H}\Delta Y) \left\{|\prod_\alpha \phi_j(q_\alpha) >\right\} \tag{B.6}$$

First, look at the second term of the Hamiltonian

$$\begin{aligned} & \left\{\prod_{\alpha'} < \phi_{j+1}(q_{\alpha'})|\right\} \sum_\beta \sum_\gamma \sum_\lambda [-V_{1\to 2}(q_\beta \to q_\gamma + q_\lambda)\hat{a}^\dagger(q_\gamma)\hat{a}^\dagger(q_\lambda)\hat{a}(q_\beta)] \left\{\prod_\alpha |\phi_j(q_\alpha) >\right\} \\ & \left\{\prod_{\alpha'} < \phi_{j+1}(q_{\alpha'})|\right\} \sum_\beta \sum_\gamma \sum_\lambda [-V_{1\to 2}(q_\beta \to q_\gamma + q_\lambda)\phi^*_{j+1}(q_\gamma)\phi^*_{j+1}(q_\lambda)\phi_j(q_\beta)] \left\{\prod_\alpha |\phi_j(q_\alpha) >\right\} \end{aligned} \tag{B.7}$$

In Eq. (B.7) we used the property of the coherent states given by Eq. (B.3).

In the continuous limit $\delta_{\alpha,\beta}$ is replaced by $\delta(q_\alpha - q_\beta)$, and $\prod_\alpha d\phi^*(q_\alpha)d\phi(q_\alpha)$ by functional integration $\int \mathcal{D}\phi^*\mathcal{D}\phi$.

From here we see that rest of the calculations is similar to that of Chapter 3.3 and we end up with the expression for a matrix element of an operator A between states of initial Y_0 and final rapidity Y

$$< Y|A|Y_0 > \sim \int \mathcal{D}\Phi^+\mathcal{D}\Phi A(Y)e^S \tag{B.8}$$

where

$$S \;=\; \int \left(\int \Phi^+(q)\frac{d}{dY}\Phi(q)dq + \mathcal{H}(1-\Phi, -\Phi^+) \right) dY \tag{B.9}$$

with the Hamiltonian given by

$$\begin{aligned}
\mathcal{H} \;=& \\
&+ \int d^4q_0 d^4q_1 d^4q_2 [V_{1\to 2}(q_0 \to q_1 + q_2)\{-\Phi(q_0) + \Phi(q_1) + \Phi(q_2) - \Phi(q_1)\Phi(q_2)\}\Phi^+(q_0) \\
&+ V_{2\to 1}(q_1 + q_2 \to q_0)\{-\Phi(q_1)\Phi(q_2) + \Phi(q_1) + \Phi(q_2) - \Phi(q_0)\}\frac{1}{2}\Phi^+(q_1)\Phi^+(q_2)]
\end{aligned} \tag{B.10}$$

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